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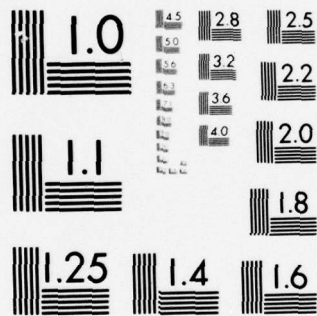
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March, 1977

A THEORY OF MIXING IN A STABLY STRATIFIED FLUID

By
Robert R. Long

Technical Report No. 11 (Series C)

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A THEORY OF MIXING IN A STABLY STRATIFIED FLUID

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ABSTRACT

A theory is developed for turbulence in a stably stratified fluid, for example in the experiments of Rouse and Dodu and of Turner. In these there is no shear and the turbulence is induced by a source of energy near the lower boundary of the fluid. A growing mixed layer of thickness D appears in the lower portion of the fluid, separated from the non-turbulent fluid above, in which the mean buoyancy gradient is given, by an interfacial layer (IL) of thickness h . The lower mixed layer has a very weak buoyancy gradient and the large buoyancy difference across IL is Δb . As indicated by the experiments of Thompson and Turner and Hopfinger and Toly, and derived by the author in a recent paper, if u is the rms horizontal velocity and ℓ is the integral length scale, $u\ell = K$ is a constant in a layer of homogeneous fluid agitated by a grid and at some distance from the grid. When there is stratification, the fluid motion is unaffected by buoyancy forces in the mixed layer so that $u\ell$ should also be constant in the lower portions of the mixed layer. Since ℓ is proportional to distance, we may conveniently suppose that the source of the disturbances is at a level $z = 0$ where u is infinite in accordance with $uz = K$. Thus we may take K to be a fundamental parameter characterizing the turbulent energy source. Then z is distance above the plane of the virtual energy source. If the upper fluid is of uniform buoyancy, $D\Delta b = v^2$ may be shown to be constant if we accept the experimental observation that h is proportional to D . In general v may be taken to be a fundamental parameter expressing the stability. The

quantity $\hat{Ri} = v^2 D^2 / K^2$ is the most fundamental of the several Richardson numbers that have been introduced in this problem because, with its use, "constants" of proportionality do not depend on the molecular coefficients of viscosity or diffusion (for high Reynolds number turbulence) or on the geometry of the grid.

The theory contains a number of results: (1) $u_e D / K \sim \hat{Ri}^{-\frac{7}{4}}$, where u_e is entrainment velocity, i. e. the rate of increase of D . This implies $D \propto t^{\frac{2}{5}}$ for a homogeneous upper fluid and $D \propto t^{\frac{1}{5}}$ for the upper linear density field. The entrainment law compares with a $-\frac{3}{2}$ law suggested by several experimenters; (2) The turbulent field in IL is intermittent with intermittency factor $I \sim \hat{Ri}^{-\frac{3}{4}}$. The turbulent patches have dimension $\delta \sim D \hat{Ri}^{-\frac{5}{4}}$; (3) If the rms horizontal velocity at some fixed level not far from the grid is denoted by u_1 , we find that the rms velocity near the interface is $u_2 \sim u_1 \hat{Ri}^{-\frac{1}{4}}$; (4) The buoyancy flux near the interface q_0 may be expressed as $q_0 \sim u_2^3 / D$ as suggested by the author in an earlier paper.

A recent experimental paper by Hopfinger and Toly proposes that $u_1 \sim u_2$. The present theory suggests that the measurements of the rms velocity were made too far from the interface.

1. Introduction.

Experimental observations beginning with those by Rouse and Dodu (1955) show that if a stably stratified fluid is agitated, say at the bottom of a container, a mixed layer develops near the bottom of depth D increasing with time. The mean buoyancy profile observed is shown schematically in figure 1. The turbulence

dies out across an interfacial layer (IL) of thickness h . The problem has great importance in meteorology and oceanography as discussed by many authors, for example Kraus and Turner (1967).

Cromwell (1960) constructed a similar experiment to simulate the pycnocline but recent interest began with the careful measurements by Turner (1968). Subsequently many others have reported on identical or similar experiments (Brush, 1970, Wolanski, 1972, Linden, 1973, Crapper and Linden, 1974, Linden, 1975, Wolanski and Brush, 1975, Thompson and Turner, 1975, Hopfinger and Toly, 1976). Other experiments involving shear have been run by Kato and Phillips (1969), Moore and Long (1971), Wu (1973), Kantha, 1975, Kantha, Phillips and Azad (1977). Typical of these is that of Kato and Phillips shown schematically in figure 2.

A number of suggestions have been made based on observation. In the experiment without shear, experimenters agree that the entrainment velocity u_e is given by¹

$$\frac{u_e}{fS} \sim Ri^{*-3/2} \quad (1)$$

where Ri^* is the overall Richardson number defined in the legend of figure 1. In addition there is strong evidence, presented by Long (1973), Crapper and Linden (1974), and more recently by Hopfinger and Toly (1976), that $h = aD$ where a is independent of the Richardson number. In Crapper and Linden's measurements h was constant as the Richardson number varied from 4 to 6000

¹ In this paper, we assume that the overall Richardson number Ri^* or similar non-dimensional number is large. If two non-dimensional numbers A, B have a ratio A/B tending to a finite non-zero constant as $Ri^* \rightarrow \infty$, we say that A is of order B and write it $A \sim B$. We use the proportionality symbol connecting two dimensional quantities that vary together.

so there can be little doubt that a is independent of Ri . It is possible, however, that a may vary with the density gradient in the upper layer.

In shearing experiments there is some indication that $u_e/u_* \sim Ri^{*-1}$ but considerable doubt has been raised by recent experiments (Kantha, 1975, Kantha, et al, 1977) in the Kato and Phillips tank.

A suggestion was made by Turner (1973) that the entrainment velocity should be expressed in terms of Δb and a velocity and length, u_1, l_1 , characteristic of the rms velocity and the integral length scale in the mixed layer. Turner suggested that $u_1 \sim fS$ and $l_1 \sim D$ and (1) implies

$$\frac{u_e}{u_1} \sim Ri_1^{-\frac{3}{2}} \quad (2)$$

where Ri_1 is the turbulent Richardson number expressed in terms of Δb , u_1 and l_1 . Long (1975), however, presented a uniform theory for all cases in which the buoyancy flux near the interfacial layer $q_0 \sim u_2^3/D$ where u_2 is the order of the velocity in the mixed layer near the interface. This implies an Ri_2^{-1} law for u_e if u_2 is the velocity used to scale u_e and in the expression for Ri_2 . Long suggested that the two results for the experiment without shear could be reconciled by assuming that the small density variation in the mixed layer was sufficient to change the order of magnitude of the turbulent velocities from $u_1 \sim fS$ in the upper part of the layer to $u_2 \sim fS Ri^{*-1/5}$ or $u_2 \propto f^{4/3}$. This was investigated experimentally by Hopfinger and Toly who concluded that the velocities were everywhere proportional to f and that the $Ri_1^{-3/2}$ law is correct. It is important for later purposes to notice that their rms velocity measurements were not made particularly close to the interface so that the observations cannot be said to disprove the theory of the author that $q_0 \sim u_2^3/D$. To avoid confusion between the paper by Long (1975) and the present paper, we remark that the present paper again finds $q_0 \sim u_2^3/D$, but that $u_2 \propto f^{3/2}$.

2. Governing Parameters.

We begin by considering the best choice of parameters governing the phenomena in the experiment. The external quantities are the frequency of oscillation of the grid f , the stroke S , lengths M_1, M_2, \dots describing the geometry and position of the grid, and the initial density variation. We assume that the dimensions of the tank are large enough to be neglected. In a recent paper (Long, 1977) the author has shown that the grid may be replaced by a virtual source of energy at a horizontal plane. The action of the source is determined by a single parameter K having the dimensions of viscosity and proportional to the constant eddy viscosity in the turbulent fluid above the source. When stratification exists, the eddy viscosity will be constant sufficiently close to the energy source since the velocities are very high there and buoyancy effects negligible as we will see. The integral length scale is proportional to the distance z from the virtual source so that if u_1 is of the order of u in the lower portion of the mixed layer at some height z_1 , we may replace K by $u_1 z_1$ when discussing the relation of experiments to theory.

The role of the density stratification may be examined by integrating the equation

$$\frac{\partial q}{\partial z} = \frac{\partial b}{\partial t} \quad (3)$$

first over the mixed layer and then over the IL. Let us assume a linear buoyancy field in the non-turbulent upper layer with buoyancy gradient N^2 . The mean buoyancy in the mixed layer is nearly constant with height and equal to

$$\bar{b}_m = b_\infty - N^2 (D+h) + \Delta b \quad (4)$$

where b_∞ is constant. We get

$$q_0 = D \frac{d\Delta b}{dt} - N^2 D \frac{d}{dt}(D+h) \quad (5)$$

In the IL, the mean buoyancy is

$$\Delta b = \frac{\Delta b}{h} (z-D) + b_{\infty} - N^2 (D+h) \quad (6)$$

Integrating (3), we get

$$q = q_0 + \frac{d\Delta b}{dt} \left(\zeta - \frac{\zeta^2}{2h} \right) + \Delta b \left(\frac{\zeta^2}{2h^2} \frac{dh}{dt} + \frac{\zeta}{h} \frac{dD}{dt} \right) - N^2 \zeta \left(\frac{dD}{dt} + \frac{dh}{dt} \right) \quad (7)$$

where $\zeta = z-D$. At $\zeta = h$ the buoyancy flux is zero. Using this and (5) we get

$$\frac{d}{dt} \left[\left(D + \frac{h}{2} \right) \Delta b - \frac{N^2}{2} (D+h)^2 \right] = 0 \quad (8)$$

If the buoyancy field at the initial instant was the linear field $b_{\infty} - N^2 z$, Eq. (8) becomes

$$\left(D + \frac{h}{2} \right) \Delta b = \frac{N^2}{2} (D+h)^2 \quad (9)$$

and N is the fundamental parameter characterizing the buoyancy field. Otherwise the constant

$$V^2 = \left(D + \frac{h}{2} \right) \Delta b - \frac{N^2}{2} (D+h)^2 \quad (10)$$

and N may be used as fundamental constants. If the upper fluid is homogeneous, $V^2 = \left(D + \frac{h}{2} \right) \Delta b$ is constant. Using $h = aD$, we obtain $D\Delta b = v^2$ is constant and v is the basic parameter characterizing the buoyancy effect.

3. The Interfacial Layer.

The energy equation at any level z is

$$0 = - \frac{\partial}{\partial z} \left[\frac{w}{2} (2p/\rho_0 + u^2 + v^2 + w^2) \right] - wb - \epsilon \quad (11)$$

where the first term is the energy flux divergence, u , v , w are the velocities, p is pressure, ρ_0 is the reference density, $q = -wb$ is the buoyancy flux and ϵ

is the energy dissipation. We must have turbulence, although perhaps intermittent, in the IL. Since the only buoyancy flux occurs in the turbulent regions, we get at any level

$$q = -A_1 u_3 b_3 I \quad (12)$$

where A_1 is a universal constant, u_3 is the order of the velocity, b_3 is the rms buoyancy fluctuation and I is the intermittency factor. The turbulence is certainly strongly influenced by buoyancy in this layer and it is reasonable to assume that kinetic and available potential energies are of the same order not only in the waves but in the turbulent patches, i. e. ,

$$u_3^2 \sim \delta b_3 \sim \delta^2 \frac{\Delta b}{h} \quad (13)$$

where δ is of the order of the dimensions of the patch, and u_3 is the rms velocity in the IL. We get

$$\frac{\delta}{h} = A_2 \frac{u_3}{(h \Delta b)^{\frac{1}{2}}} \quad (14)$$

Using (13), equation (12) becomes

$$q = -A_3 \frac{u_3^3 I (h \Delta b)^{\frac{1}{2}}}{h u_3} \quad (15)$$

where A_2 and A_3 are universal constants. Let us now compute the dissipation. This occurs only in the turbulent patches so that

$$\epsilon \sim I \frac{u_3^3}{\delta} \sim \frac{u_3^3}{h} I \frac{(h \Delta b)^{\frac{1}{2}}}{u_3} \quad (16)$$

so that $\epsilon \sim q$. Since they are both dissipative, it follows that they are of the order of the energy flux divergence. At the upper boundary of the IL, the kinetic energy of the waves has been so reduced by losses to potential energy and dissipation, that there can no longer be wave breaking and turbulence.

Thus h is the depth of penetration of the turbulence. When the upper fluid has a uniform density, the motion above is irrotational and, neglecting the very small time changes of kinetic energy, the energy flux must be zero just above $z = D+h$ and therefore zero just below. Then the increment of energy flux across the IL is equal to the energy flux into the bottom of the layer. If there is a density gradient in the fluid above the IL, energy flux may exist there because of the existence of internal gravity waves. At the height $z = D + h$, however, the energy flux is too weak to support turbulence so that it has decreased to a value well below that at the bottom of the IL. Therefore, in all cases, the increment in energy flux is proportional to the value at the bottom of the IL. The constant of proportionality will vary with the density variation in the upper layer. Integrating between levels just outside of the layer, we get for the mean buoyancy flux

$$\bar{q} h \sim u_2^3 \quad (17)$$

where u_2 is the rms velocity in region R_2 just outside the IL. Obviously since pressures are continuous, there can be no change of order of magnitude¹ of the velocity across the level $z = D$ so that $u_2 \sim u_3$ at $z = D$. Using Eq. (7) we may write

$$-\frac{c_1 u_2^3}{h} = D \frac{d\Delta b}{dt} + \frac{h}{3} \frac{d\Delta b}{dt} + \frac{\Delta b}{6} \frac{dh}{dt} + \frac{1}{2} \Delta b \frac{dD}{dt} - N^2 \left(D + \frac{h}{2} \right) \frac{d}{dt} (D+h) \quad (18)$$

¹ More carefully, we recognize that the bottom of the IL is in wave motion with occasional small areas of breaking. The front between the turbulence below and the mostly laminar flow in the IL is very sharp. Instantaneously the pressures on this surface are of order u_2^2 on the lower side and therefore u_2^2 on the upper side. The latter corresponds to velocities of order $u_2 \sim u_3$ in the lower portion of the IL.

$$\frac{\delta_0}{h} = A_2 \frac{u_2}{(h\Delta b)^{\frac{1}{2}}} \quad (19)$$

$$A_3 I_0 u_2^2 \left(\frac{\Delta b}{h} \right)^{\frac{1}{2}} = -D \frac{d\Delta b}{dt} + N^2 D \frac{d}{dt} (h+D) \quad (20)$$

where I_0 and δ_0 are evaluated at a level just above $z = D$ and c_1 is a constant of order one which may vary with the stability of the upper layer.

It is of interest to compute the energy flux in the lower part of the IL. In the eddies the contribution is of order Iu_2^3 . Thus the transfer is due primarily to the waves and is of order of $u_2 p \sim u_2^2 c$ where c is the speed of the energy containing waves of length λ , i. e., $c \sim \lambda \left(\frac{\Delta b}{h} \right)^{\frac{1}{2}}$. Since $u_2^2 c$ is of the order of u_2^3 , $c \sim u_2$ or

$$\lambda \sim u_2 \left(\frac{h}{\Delta b} \right)^{\frac{1}{2}} \quad (21)$$

This suggests that the eddies in the region R_2 of length λ are resonating with these waves. Since λ is of the order of both wave length and amplitude of the breaking waves, then δ_0 should be of order λ . Comparison of (19) and (21) shows that this is indeed the case. The bottom surface of the interfacial layer will have waves of length and amplitude δ_0 so that the layer R_2 may be considered to be of thickness δ_0 .

4. Turbulence in the Mixed Layer and Final Results.

So far we have simply assumed turbulence below the IL without considering its properties in detail and the relation of these to the flux q_0 or to the entrainment velocity u_e . Let us consider the effect of buoyancy in the R_2 layer where the magnitude of u is u_2 . Here $q \sim u_2 b_2 \sim \frac{u_2^3}{h}$. Thus $b_2 h \sim u_2^2$ so that the ratio of

kinetic to potential energy of the eddies is

$$\frac{u_2^2}{b_2 \delta} \sim \frac{h}{\delta} \quad (22)$$

This is very large, so that the buoyancy variation is unimportant dynamically in R_2 . This will also be true in the mixed layer as a whole so that in the whole mixed layer we may assume that the energy dissipation function is determined by factors independent of the buoyancy. The dissipation function at any height z in the mixed layer must depend only on K , z and D , i. e., $u^3/\ell = f(K, z, D)$ where ℓ is the eddy length. Thus $u^3 z^4/\ell K^3 = f(z/D)$. At height $z = D - \delta_0 \approx D$, the length ℓ is of order δ_0 and the velocity is u_2 so that

$$\frac{u_2^3 D^4}{\delta_0 K^3} = A_u \quad (23)$$

where A_u is a universal constant. Using (18)-(20) we may write

$$\frac{d}{dt} \left[\left(D + \frac{h}{2} \right) \Delta b - \frac{N^2}{2} (D+h)^2 \right] = 0 \quad (24)$$

$$-\frac{cK^{\frac{3}{2}}}{h^{\frac{1}{4}} D^6 (\Delta b)^{\frac{3}{4}}} = D \frac{d\Delta b}{dt} + \frac{h}{3} \frac{d\Delta b}{dt} + \frac{\Delta b}{6} \frac{dh}{dt} + \frac{1}{2} \Delta b \frac{dD}{dt} - N^2 (D + \frac{h}{2}) \frac{d}{dt} (D+h) \quad (25)$$

$$\frac{\delta_0}{D} = A_4 \frac{K^{\frac{3}{2}}}{D^3} \left(\frac{h}{\Delta b} \right)^{\frac{3}{4}} \quad (26)$$

$$-A_5 I_0 \frac{K^3}{D^4} = D \frac{d\Delta b}{dt} - N^2 D \frac{d}{dt} (h+D) \quad (27)$$

where A_4 and A_5 are universal constants and c is a constant of order one, varying with the buoyancy gradient in the upper layer.

We now make the assumption based on the observations discussed in Section 1 that the thickness of the interfacial layer is proportional to D and

is independent of the stability. When the upper fluid is homogeneous, $D\Delta b = v^2$ is constant and in terms of the fundamental Richardson number $\hat{Ri} = v^2 D^2 / K^2$ we obtain

$$\frac{u_a D}{K} = \alpha_1 \hat{Ri}^{-\frac{7}{4}} \quad (28)$$

$$I_0 = \alpha_2 \hat{Ri}^{-\frac{3}{4}} \quad (29)$$

$$\frac{\delta_0}{D} = \alpha_3 \hat{Ri}^{-\frac{3}{4}} \quad (30)$$

$$\frac{u_2 D}{K} = \alpha_4 \hat{Ri}^{-\frac{1}{4}} \quad (31)$$

where α_i are constants. As shown by the author (Long, 1977), u_1 (or K) is proportional to the grid frequency f so that Eq. (28) leads to

$$\frac{u_a}{fS} \sim Ri^{*- \frac{7}{4}}, \quad Ri^* = \frac{D\Delta b}{f^2 S^2} \quad (32)$$

compared to $Ri^{*- \frac{3}{2}}$ suggested by experimenters. Inspection of their data, however, reveals little reason to choose one law in preference to another. Notice that Eq. (28) may be integrated to yield

$$D \propto t^{\frac{2}{11}}$$

The results in (28)-(31) are the same for the erosion of a linear density gradient where, however, the constants may vary with the stability of the upper layer. The entrainment law now yields $D \propto t^{\frac{1}{9}}$.

5. Summary and Conclusions.

It is now possible to form a reasonable description of the state of affairs in the experiment. The oscillating grid is nearly equivalent to a source of energy on a plane ($z = 0$) as discussed by the author (1977). A single parameter K

characterizes the source where K has dimensions $L^2 T^{-1}$. If the fluid is homogeneous, the source started at $t = 0$, causes a turbulent layer to develop of depth $D \sim (Kt)^{\frac{1}{2}}$ separated by a front from the non-turbulent fluid above. Thus the front propagates at a speed proportional to $t^{-\frac{1}{2}}$. After the front has moved far away, conditions ultimately reach a steady state with energy supplied by the source. In any layer there is a balance between the energy entering at the lower plane less the energy leaving through the upper plane and the energy dissipation. The rms velocity components are proportional to each other with universal constants of proportionality. The horizontal component, for example, is given by

$$\frac{uz}{K} = B_1 \left(\frac{K}{\nu} \right) \quad (33)$$

and the integral length scale by

$$\frac{\ell}{z} = B_2 \left(\frac{K}{\nu} \right) \quad (34)$$

where ν is the molecular viscosity. The energy dissipation ϵ decreases rapidly with distance from the grid as K^3/z^4 . If a passive additive is present, its mean concentration \bar{S} is determined by solutions of the equation

$$\frac{\partial \bar{S}}{\partial t} = K_h \frac{\partial^2 \bar{S}}{\partial z^2} \quad (35)$$

where K_h is the constant eddy diffusivity proportional to K .

If the fluid is initially stratified, the energy source creates a mixed layer of thickness D , increasing much more slowly with time, separated from the non-turbulent fluid above by an interfacial layer of thickness h . Although h is small compared with D , it remains proportional to D . The mixed layer has a very weak buoyancy gradient so that dissipation nearly balances the

energy flux divergence, and the loss of kinetic energy to potential energy is small. This is reflected in the fact that the kinetic energy u^2 is much greater than the available potential energy ρb where ℓ is the eddy scale and b is the perturbation buoyancy. The eddy length ℓ is proportional to distance from the source, becoming of order D in the center of the layer. ℓ decreases again toward the interface becoming of order $\delta_0 \sim D \hat{Ri}^{-\frac{3}{4}}$.

The rms velocity u varies as K/z in the upper part of the mixed layer, decreasing with z but maintaining a proportionality to K . In the experiment, if viscosity is negligible K is proportional to the frequency of the grid f . As we approach the interface the eddy size decreases ultimately to δ_0 in the vicinity of the interface which is disturbed by waves of length and amplitude of order δ_0 . Since the energy dissipation is of order u^3/ℓ , the dissipation tends to increase by the reduction of ℓ but this is offset by the loss of kinetic energy to heat which reduces u . If the stability is infinitely great, i. e. $(K^2/v^2 D^2) \rightarrow 0$, the rms velocity tends to zero at the interface. For finite but small values of K/vD , we have $u_2 \sim (K/D) \hat{Ri}^{-\frac{1}{4}}$. This indicates $u_2 \propto f^{\frac{3}{2}}$ instead of $f^{\frac{4}{3}}$ as originally suggested by the author. The difference lies in the mechanism for the decrease of u as we approach the interface. The author's original suggestion was that the buoyancy gradient, although weak in the mixed layer, was sufficiently strong to decrease the kinetic energy at the expense of potential energy. In the present theory u decreases because of dissipation to heat as the eddy size decreases near the interface. However, u does not go to zero but to a value corresponding to a small but non-zero eddy length δ_0 , proportional to $\hat{Ri}^{-\frac{3}{4}}$. This dimension is also the dimension of the layer over which the interface moves up and down.

This length was measured by Hopfinger and Toly as proportional to Ri^{*-1} where Ri^* is proportional to $\hat{R}i$.

The interfacial layer is very stable and has intermittent turbulence with intermittency factor $I \sim \hat{R}i^{-\frac{3}{4}}$. The turbulence in patches of dimensions δ_0 is caused by breaking waves of length and amplitude δ . These are energized by resonance with the pressure fluctuations on the lower interfacial surface of frequency u_2/δ_0 equal to the natural frequency $N = (\Delta b/h)^{\frac{1}{2}}$. Equating these and putting $h \sim D$ leads to $\delta_0 \sim D \hat{R}i^{-\frac{3}{4}}$ as in Eq. (30).

Finally the buoyancy flux varies in both the mixed layer and the interfacial layer. It is a maximum q_0 at the level $z = D$ where it is of order u_2^3/D determined by the rms velocity in the mixed layer close to the interface as suggested by the author in an earlier paper. $q_0 \sim u_e \Delta b$ where u_e is the entrainment velocity or rate of increase of D and is proportional to $\hat{R}i^{-\frac{7}{4}}$. This is close to measurements, but differs slightly from the $Ri^{*- \frac{3}{2}}$ law proposed by the experimenters. The entrainment law leads to a variation of D as $t^{\frac{2}{1+4}}$ compared to $D \propto t^{\frac{1}{5}}$ when the upper fluid has a linear density gradient. In the case of a linear gradient in the upper fluid the results are as stated above except that the constants of proportionality may vary with the stability of the upper fluid. The $t^{\frac{1}{5}}$ law reveals a slower rate of increase for the linear case as expected but it is somewhat slower than that measured by Linden (1975).

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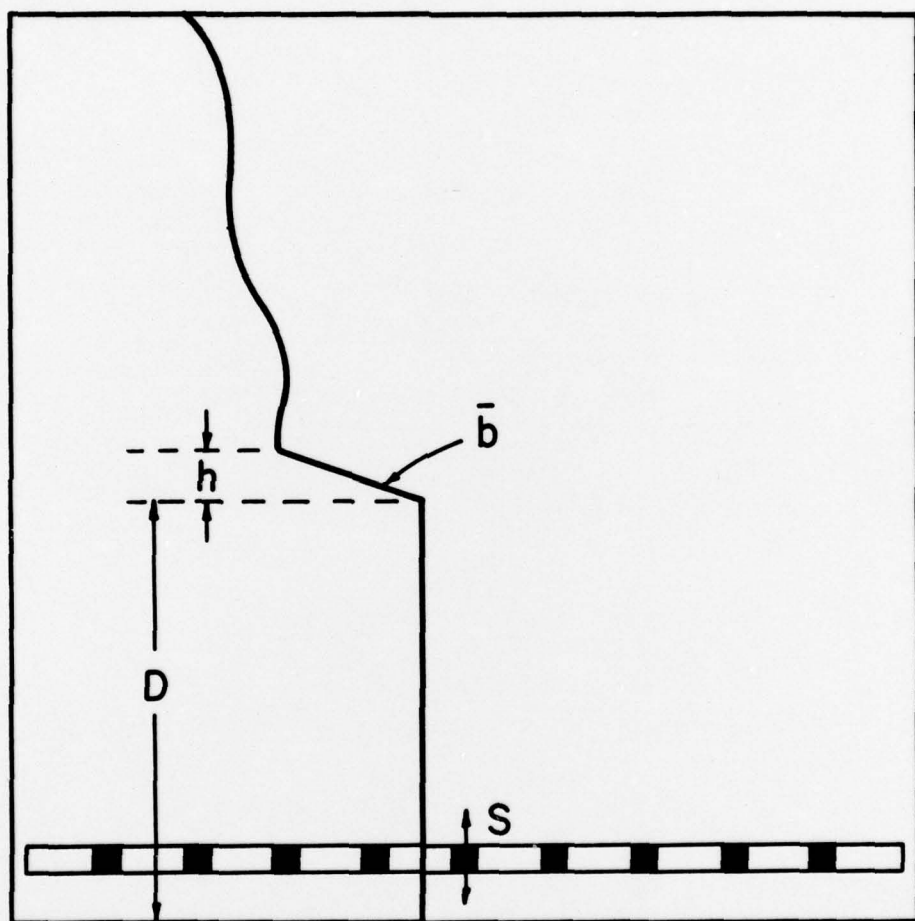
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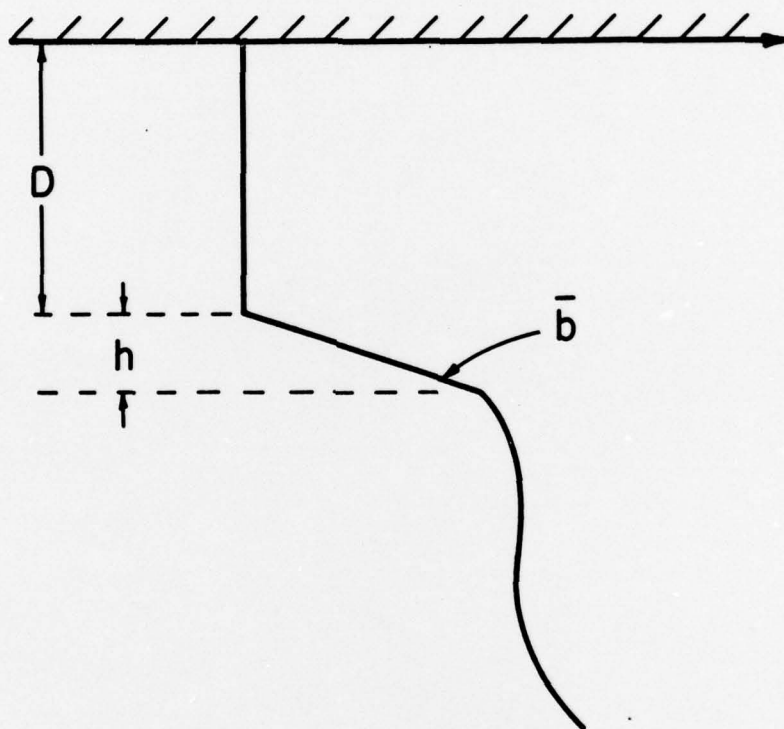
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LEGENDS

Figure 1. The container has a fluid with variable mean buoyancy profile. The lower layer of depth D is fully turbulent and has a weak buoyancy gradient. The buoyancy decreases strongly in an interfacial layer (IL) of thickness h between the mixed layer and the non-turbulent layer above, the buoyancy difference across h being Δb . The upper layer has a buoyancy gradient $\bar{b}(z)$. The turbulence is caused by a grid oscillating up and down with stroke S and frequency f . The geometry and location of the grid is given by lengths M_1, M_2, \dots . An important non-dimensional number is $Ri^* = D\Delta b/f^2 S^2$.

Figure 2. Kato and Phillips experiment with mean buoyancy profile. The turbulence in the upper layer is caused by a screen moving along the surface exerting a stress $\tau = u_*^2$, where u_* is the friction velocity. An important non-dimensional number is $Ri^* = D\Delta b/u_*^2$.





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4. ABSTRACT <p> A theory is developed for turbulence in a stably stratified fluid, for example in the experiments of Rouse and Dodu and of Turner. In these there is no shear and the turbulence is induced by a source of energy near the lower boundary of the fluid. A growing mixed layer of thickness D appears in the lower portion of the fluid, separated from the non-turbulent fluid above, in which the mean buoyancy gradient is given, by an interfacial layer (IL) of thickness h. The lower mixed layer has a very weak buoyancy gradient and the large buoyancy difference across IL is Δb. As indicated by the experiments of Thompson and Turner and Hopfinger and Toly, and derived by the author in a recent paper, if u is the rms horizontal velocity and l is the integral length scale, $ul = K$ is a constant in a layer of homogeneous fluid agitated by a grid and at some distance from the grid. When there is stratification, the fluid motion is unaffected by buoyancy forces in the mixed layer, so that ul should also be constant in the lower portions of the mixed layer. Since l is proportional to distance, we may conveniently suppose that the source of the disturbances is at a level $z = 0$ where u is infinite in accordance with $uz = K$. Thus we may take K to be a fundamental parameter characterizing the turbulent energy source. Then z is distance above the plane of the virtual energy source. If the upper fluid is of uniform buoyancy, $D\Delta b = v^2$ may be shown to be constant if we accept the experimental observation that h is proportional to D. In general v may be taken to be a fundamental parameter expressing the stability. The quantity $Ri = v^2 D^3 / K^2$ is the most fundamental of the several Richardson numbers that have been introduced in this problem because, with its use, "constants" of proportionality do not depend on the molecular coefficients of viscosity or diffusion (for high Reynolds number turbulence) or on the geometry of the grid. </p> <div style="display: flex; justify-content: space-between; margin-top: 10px;"> $D\Delta b = v^2$ squared $(vD/K)^2$ squared (continued over) </div>			
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13. Abstract (continuation)

The theory contains a number of results: (1) $u_e D/K \sim \hat{Ri}^{-\frac{7}{4}}$, where u_e is entrainment velocity, i.e. the rate of increase of D . This implies $D \propto t^{\frac{2}{3}}$ for a homogeneous upper and $D \propto t^{\frac{1}{3}}$ for the upper linear density field. The entrainment law compares with a $-\frac{3}{2}$ suggested by several experimenters; (2) The turbulent field in IL is intermittent with intermittency factor $I \sim \hat{Ri}^{-\frac{3}{4}}$. The turbulent patches have dimension $\delta \sim D \hat{Ri}^{-\frac{3}{4}}$; (3) If the rms horizontal velocity at some fixed level not far from the grid is denoted by u_1 , we find the rms velocity near the interface is $u_2 \sim u_1 \hat{Ri}^{-\frac{1}{4}}$; (4) The buoyancy flux near the interface may be expressed as $q_0 \sim u_2^3/D$ as suggested by the author in an earlier paper.

A recent experimental paper by Hopfinger and Toly proposes that $u_1 \sim u_2$. The present theory suggests that the measurements of the rms velocity were made too far from the interface.